Introduction to higher topoi

2: Descent into ∞-topoi

Charles Rezk
(U. of Illinois, Urbana-Champaign)

<u>**Grothendieck topos**</u>: category $\mathcal{E}$ such that $\exists$

(1) small category C

(2) fully faithful $\mathcal{E} \xrightarrow{i} P(C) := \mathrm{Fun}(C^{op}, \mathrm{Set})$, with

(3) a left adjoint $\ell \dashv i$, s.t.

(4) ℓ preserves finite limits (left exact)

Correspondence:

$$\left(\begin{array}{c}\text{lex localizations of}\\ \mathrm{Fun}(C^{op}, \mathrm{Set})\end{array}\right) \Longleftrightarrow \left(\begin{array}{c}\text{Grothendieck}\\ \text{topologies on } C\end{array}\right)$$

"well-known"

Basic properties of Gr. Topoi : follow immediately

- **Cartesian closure**: $[\![X,Y]\!] \in P(C) \underset{}{\overset{\ell}{\rightleftarrows}} \mathcal{E}$ $\qquad \ell(X \times Y) \cong \ell X \times \ell Y$

  $Y \in \mathcal{E} \implies [\![X,Y]\!] \in \mathcal{E}$

- **Slices**: $P(C) \underset{i}{\overset{\ell}{\rightleftarrows}} \mathcal{E} \ni X = \ell X \implies P(C)_{/X} \underset{i}{\overset{\ell}{\rightleftarrows}} \mathcal{E}_{/X}$ – lex

  $P(C)_{/X} = P(C/X)$

- **Subobject classifier**: $\Omega \in P(C) \overset{\ell}{\rightleftarrows} \mathcal{E}$ $\quad \checkmark$

  $\ell \implies \Omega \xrightarrow{j} \Omega$ idempotent $\implies j\Omega \in \mathcal{E}$

<u>∞-topos</u>: ∞-category $\mathcal{E}$ such that $\exists$

(1) small ∞-category C

(2) accessible + fully faithful $\mathcal{E} \xhookrightarrow{i} \underline{PSh(C)} = Fun(C^{op}, \underline{\underline{\underline{S}}})$, with

presheaf of ∞-groupoids

(3) a left adjoint $\ell \dashv i$, s.t.

(4) ℓ preserves finite limits ("left exact")

"left exact localization of presheaves"

accessible functor $F : C \to D$ of ∞-categories:

- C, D cocomplete
- F preserves κ-filtered colimits (some regular cardinal κ)

- J is κ-filtered if

$$\begin{tikzcd} I \arrow[r] \arrow[d, hook] & J \\ I \star \Delta^0 \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

$\forall$ κ-small ∞-cats I

In Gr-topos accessibility of j follows from other properties

Lurie, "HTT", Ch. 5 $\Leftarrow$

Presentable ∞-category: $\mathcal{C}$ such that $\exists$

(1) small ∞-category C

(2) accessible + fully faithful $\mathcal{C} \xrightarrow{i} \mathrm{PSh}(C) = \mathrm{Fun}(C^{op}, \mathcal{S})$, with

(3) a left adjoint $\ell \dashv i$

∞-cat generalization of "locally presentable categories"

Lurie, "HTT", Ch 5. [also Simpson (arXiv 1999)]

What about Grothendieck topologies?

· Given (C, τ) (Grothendieck site on a 1-category C)

$$\Longrightarrow \quad \underbrace{Sh(C,\tau)}_{\infty\text{-topos}} \subseteq PSh(C) := Fun(C^{op}, \mathcal{S})$$

<u>sheaves</u>

Ex: X = top'l space. $\Longrightarrow$ $Sh(X) \subseteq Fun(Open_X^{op}, \mathcal{S})$

defined in previous hour

Characterize Grothendieck topoi

Giraud Theorem: a 1-category $\mathcal{E}$ is a Grothendieck topos iff

- $\mathcal{E}$ is locally presentable.
- colimits are universal
- coproducts are disjoint
- equivalence relations are effective.

Characterize ∞-topoi:

Theorem: An ∞-category $\mathcal{E}$ is an ∞-topos iff

- $\mathcal{E}$ is presentable
- colimits are universal
- colimits satisfy descent

"descent"

have roots in homotopy

Homotopy limits/colimits : in top'l spaces

$$\begin{tikzcd}
U \cap V \arrow[r, hook] \arrow[d, hook] & U \arrow[d, hook, "\text{open}"] \\
V \arrow[r, hook] & X = U \cup V \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

$\Rightarrow$ behaves well wrt H_*

$$\begin{tikzcd}
F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p", "\text{fibre bundle}"'] \\
\{b\} \arrow[r] & B
\end{tikzcd}$$

$\Rightarrow$ Behaves well wrt π_*

Homotopy pushouts/pullbacks:

In Top or sSet [or (proper) QMC]

"cofibration"

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail] & A' \arrow[d, tail] \\ B \arrow[r] & B' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

homotopy pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & E' \arrow[d, two heads] \\ B' \arrow[r] & B \end{tikzcd}$$

"fibration"

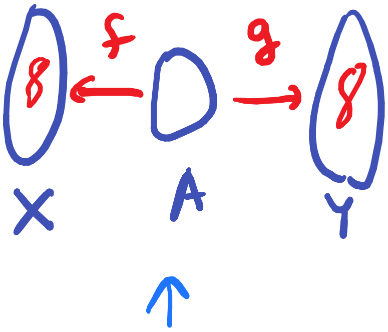
homotopy pullback

To "compute" homotopy pushout:

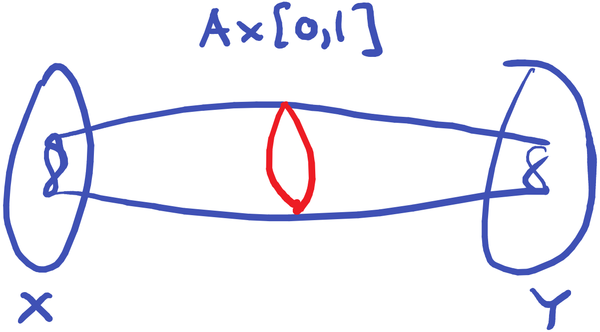
$$\begin{tikzcd} A \arrow[rr] \arrow[dr, tail, "\text{cof}"'] & & B \\ & \tilde{B} \arrow[ur, "\sim", "\text{w.e.}"'] & \end{tikzcd} \Rightarrow \operatorname{colim}\left(\begin{tikzcd} A \arrow[r] \arrow[d, tail] & A \arrow[d] \\ \tilde{B} \arrow[r] & \tilde{B}' \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \right)$$

Get nice pictures:

$$\tilde{X} \leftarrow A \rightarrow \tilde{Y}$$



$\rightsquigarrow$



"double mapping cylinder"

homotopy pushout

Homotopy limits/colimits

are "derived functors":

<u>best</u> homotopy invariant approximation to lim/colim in Top/sSet (or in Q.M.C.).

$\Rightarrow$ In ∞-categories ($\Leftarrow$ "limits/colimits")

Spaces are special:

(homotopy theory of Top, or sSet $\Rightarrow \mathcal{S}$)

Homotopy limits/colimits here have additional properties, not satisfied in general. What are these?

$\Rightarrow$ "∞-topos" arises from one answer to this

Set $\rightsquigarrow$ topos $\qquad\qquad$ $\mathcal{S} \rightsquigarrow \infty$-topos

Universality of (homotopy) pushouts: in sSet

homotopy pushout is always equiv to

in sSet cof = mono

$$\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[d, tail] & X_1 \arrow[d, tail] & & Y_0 \arrow[r, dashed] \arrow[d, dashed] & Y_1 = p^{-1}X_1 \arrow[d, dashed] \\
X_2 \arrow[r, tail] & X \arrow[ul, phantom, "\ulcorner", very near start] & & Y_2 \arrow[r] & Y \arrow[lll, bend left, "p"]
\end{tikzcd}$$

monos

Universality of (homotopy) pushouts: in sSet

homotopy pushout is always equiv to

$$Y_1 := X_1 \times_X Y$$

$$\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[d, tail] \arrow[dr, phantom, "\text{ho.p.o}"] & X_1 \arrow[d] & & Y_0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\text{ho.po.}"] & Y_1 \arrow[d] \\
X_2 \arrow[r, tail] & X & & Y_2 \arrow[r, tail] & Y \arrow[lll, bend left, "p"]
\end{tikzcd}$$

monos

pullback

"pushouts are universal in sSet"

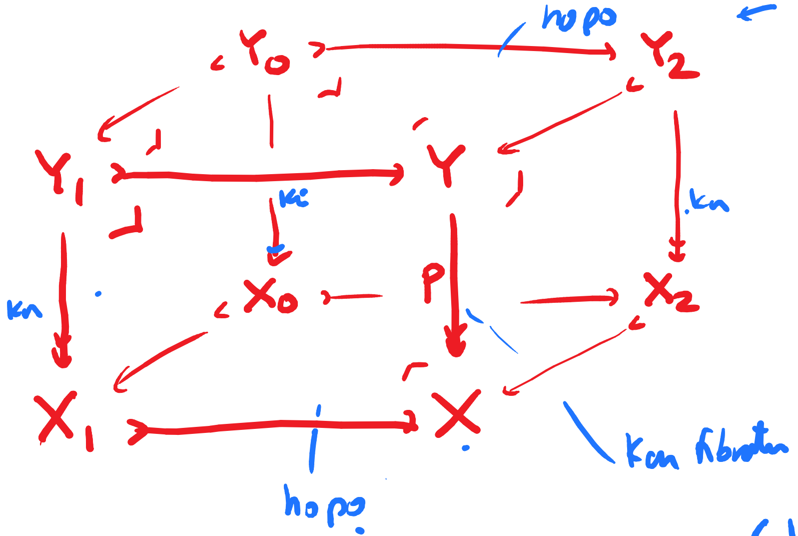
In any 1-topos $\mathcal{E}$ (e.g. sSet):

colimits are universal: $\forall \; f : Y \to X$ in $\mathcal{E}$:

$$f^* : \mathcal{E}_{/X} \longrightarrow \mathcal{E}_{/Y} \text{ preserves } \underline{\text{colimits}}$$

(have right adjoint $\Pi_f : \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$)

If $p: Y \to X$ is also a <u>Kan fibration</u>,



all 4 sides are ho pbs.

"ho pushouts are universal in homotopy theory of sSet"

(also ho. colimit)

Descent for homotopy pushouts:

"opposite composition of operations"

$$\begin{tikzcd}
Y_1 \arrow[d] \arrow[dr, phantom, "\text{ho pb}"] & Y_0 \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{ho pb}"] & Y_2 \arrow[d] & \overset{\text{ho colim}}{\Longrightarrow} & Y \arrow[d] \\
X_1 & X_0 \arrow[l] \arrow[r] & X_2 & \underset{\text{ho colim}}{\Longrightarrow} & X
\end{tikzcd}$$

$\rightsquigarrow$

Descent:

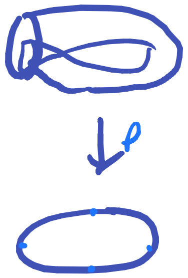
$$\begin{tikzcd}
Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\
X_i \arrow[r] & X
\end{tikzcd}$$

is a homotopy pullback $i = 0, 1, 2$.

Example: $f: X \to X$ (in Top) (or sSet)

$$\begin{tikzcd} X \arrow[d] & X \sqcup X \arrow[l, "(id,id)"'] \arrow[r, "(id,f)"] \arrow[d] \arrow[dl, phantom, "\text{ho pb}"] \arrow[dr, phantom, "\text{ho pb}"] & X \arrow[d] \\ * & \{1,2\} \arrow[l] \arrow[r] & * \end{tikzcd} \quad \xRightarrow{\text{ho po}} \quad \begin{tikzcd} C_f \arrow[d] \\ S^1 \end{tikzcd}$$

"mapping cylinder"



$$C_f = X\times[0,1] / (x,1) \sim (fx,0)$$

f homeomorphism : p fiber bundle, fibers $\simeq X$

f homotopy equiv : not fiber bundle, but ho fibers $\underset{\text{w.e.}}{\simeq} X$

Not true in Set (or any Gr topos): $f : X \xrightarrow[\text{iso}]{\cong} X$

$$\begin{tikzcd}
X \arrow[d] & X \sqcup X \arrow[l, "{(id,id)}"'] \arrow[r, "{(id,f)}"] \arrow[d] & X \arrow[d] & \xrightarrow{\text{colim}} & X/_{x \sim f(x)} \\
* & \{1,2\} \arrow[l] \arrow[r] & * & \xrightarrow{\text{colim}} & *
\end{tikzcd}$$

$? = 1, 2$

$\forall$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "?"] & X/\sim \arrow[d] \\
* \arrow[r] & *
\end{tikzcd}$$

p.b only if $f = 1_X$

[can work if

e.g. $\leftarrowtail \; \rightarrowtail$]

Earliest statement I can find is in

G. Segal, "Categories and cohomology theories" (1974, preprint 1969)

"well-known"

proof sketch in

Puppe, "Remark on 'homotopy fibrations'" (1974)

independent proof in

Mather, "Pullbacks in homotopy theory" (1976)

Sketch of proof in sSet model of homotopy theory

ho pb

$$\begin{tikzcd} Y_1 \arrow[d, "fib"'] & Y_0 \arrow[l] \arrow[r] \arrow[d, "fib"] & Y_2 \arrow[d, "fib"] \\ X_1 & X_0 \arrow[l, "cof"] \arrow[r, "cof"'] & X_2 \end{tikzcd} \xRightarrow[(\simeq \text{ho colim})]{\text{colim}} \begin{tikzcd} Y \arrow[d] \\ X \end{tikzcd}$$

Is

$$\begin{tikzcd} Y_j \arrow[r] \arrow[d] & Y \arrow[d] \\ X_j \arrow[r] & X \end{tikzcd}$$

ho pb ?

Reduce to case where indicated maps are cof / fib

Special case which works: squares ho. pb and p.b

Factor homotopy pullback through one which is actual pullback: (along mono)

$$
\begin{tikzcd}
Y_0 \arrow[r, dashed] \arrow[d, "\mathrm{w.e.}"] \arrow[dd, bend right=60, "\mathrm{Kan}"'] & \boxed{\Pi_j Y_0} \arrow[d, "\mathrm{w.e.}"] \arrow[dd, bend left=60, "\mathrm{Kan}"] \\
j^*Y_1 \arrow[r, tail, "j"] \arrow[d, "\mathrm{Kan}"] & Y_1 \arrow[d, "\mathrm{Kan}"] \\
X_0 \arrow[r, tail, "j"] & X_1
\end{tikzcd}
$$

<u>Recall</u>: $j : A \rightarrowtail B$ in sSet

$$
\Rightarrow \quad
\begin{tikzcd}
sSet_{/B} \arrow[r, "j^*"] & sSet_{/A} \arrow[l, bend right, "\Sigma_j"'] \arrow[l, bend left, "\Pi_j"]
\end{tikzcd}
$$

j mono $\Rightarrow$ $j^* \circ \Pi_j \cong \mathrm{id}$

<u>Miracle</u>: works!

Kapulkin - Lumsdaine (Voevodsky) "The Simplicial model of Univalent Foundations"

Universality of colimits

in ∞-category $\mathcal{E}$

(cocomplete, finite complete):

For all morphisms $f : X \to Y$ induced pullback

$f^* : \mathcal{E}_{/Y} \to \mathcal{E}_{/X}$ preserves colimits

Natural transformation $\lambda : F \Rightarrow F'$ of $F, F' : \mathcal{I} \to \mathcal{E}$

is **Cartesian** if $\forall$ $\alpha : i \to j$ in $\mathcal{I}$,

$$\begin{tikzcd} F(i) \arrow[r, "\lambda"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & F'(i) \arrow[d, "\alpha"] \\ F(j) \arrow[r, "\lambda"'] & F'(j) \end{tikzcd}$$

pullback in $\mathcal{E}$.

$$\mathcal{E}^{\rightarrow} := \mathrm{Fun}(\underset{(\cdot \to \cdot)}{\underset{\shortparallel}{\Delta^1}}, \mathcal{E})$$ "arrow category"

$$\mathrm{Cart}(\mathcal{E}^{\rightarrow}) \hookrightarrow \mathcal{E}^{\rightarrow}$$ subcategory (wide, not full) of Cartesian transformations $\lrcorner$

$\mathcal{E}$ has **descent** if

- $\mathrm{Cart}(\mathcal{E}^{\rightarrow})$ has colimits
- $\mathrm{Cart}(\mathcal{E}^{\rightarrow}) \to \mathcal{E}^{\rightarrow}$ preserves colimits

Puppe (1974).

Theorem: Every ∞-topos has universal colimits + descent

Proof: (1) $\mathcal{S}$ ✓ (uses sSet model).

(2) $\rightsquigarrow$ $\mathrm{PSh}(C) \simeq \mathrm{Fun}(C^{op}, \mathcal{S})$ lim/colim computed pointwise

(3) $\mathrm{PSh}(C) \underset{i}{\overset{\ell \text{ lex}}{\rightleftarrows}} \mathcal{E}$. ℓ, i preserve finite lim / all colims

Theorem: An ∞-category $\mathcal{E}$ is an ∞-topos iff

- $\mathcal{E}$ is presentable
- colimits are universal
- descent

Sketch $\Leftarrow$

(1) Find essentially small $\mathcal{C} \subseteq \mathcal{E}$,

· closed under finite limits, s.t.

$$\mathcal{E} \xrightarrow{i} \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$$ is · fully faithful and (i accessible)

restricted yoneda

· has $\underline{\underline{\ell}} \dashv i$,

application of "accessible ∞-categories" (Lurie)

e.g $\mathcal{C} = \mathcal{E}^{\kappa} \subseteq \mathcal{E}$

"κ-compact" objects ("κ-presentable")

(2) <u>**Prop**</u>: $\mathcal{E}$ - cocomplete, finite complete ∞-category
universal colimits and descent.

C - small, finite complete.

$$C \underset{\text{Yoneda}}{\xhookrightarrow{e}} \mathrm{Psh}(C) \xrightarrow{\ell} \mathcal{E}, \quad \underline{\underline{\ell}} \text{ colim preserving}$$

Then

$$\ell \text{ lex} \iff \ell \circ p \text{ lex}$$

also true, $\mathcal{E} =$ 1-topos

$$\mathrm{Fun}(C^{op}, \mathrm{Set}) \xrightarrow{\ell} \mathcal{E}$$

Follows from:

Prop': $\mathcal{E}$ · cocomplete, finite complete ∞-category, universal colimits and descent.

· C · small

(not finite complete)

$C \hookrightarrow{e} \mathrm{Psh}(C) \xrightarrow{\ell} \mathcal{E}$, ℓ colim preserving

Then

ℓ lex $\Longleftrightarrow$

(1) $\ell(*) \simeq *$

(2) ℓ preserves pullbacks of form

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \rho(C_2) \arrow[d] \\ \rho(C_1) \arrow[r] & \rho(C_0) \end{tikzcd}$$

Prop' does <u>not</u> work for $\mathrm{Fun}(C^{op}, \mathrm{Set}) \xrightarrow{\ell} \mathcal{E}$ ← 1-topos

Example: G-group $\not\cong 1$ $\qquad \mathcal{E} = \mathrm{Set}$

$$\ell = \mathrm{colim} : \mathrm{Fun}(G^{op}, \mathrm{Set}) \longrightarrow \mathrm{Set}$$

hypotheses are ✓

$$\mathrm{colim}_G \left(\begin{tikzcd} G \times G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d] \\ G \arrow[r] & * \end{tikzcd} \right) = \begin{tikzcd} G \arrow[r] \arrow[d, dash] & * \arrow[d, dash] \\ * \arrow[r] & * \end{tikzcd}$$

not a pullback

Lemma:
- $\mathcal{E}$ - cocomplete, finite complete ∞-category universal colimits
- C - small

$$C \xrightarrow{e} \mathrm{Psh}(C) \xrightarrow{\ell} \mathcal{E}, \quad \ell \text{ colim preserving}$$

Then
$$\ell(X \times Y) \simeq \ell X \times \ell Y \quad \text{iff} \quad \ell(p(c) \times p(c')) \simeq \ell(p(c)) \times \ell(p(c'))$$

(also works if $\mathcal{E}$ Htopos: $\operatorname{colim}_i X_i \times \operatorname{colim}_j Y_j \simeq \operatorname{colim}_{i,j} X_i \times Y_j$)

Special Case :

$\ell : PSh(C) \longrightarrow \mathcal{E}$ colim preserving

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ X \arrow[r] & B \end{tikzcd} \in Fun(I, PSh(C)) \overset{\ell}{\Longrightarrow} \text{p.b ?}$$

$B \simeq p(c)$

Same as :

product in
$\Downarrow$

$PSh(C_{/c}) \simeq PSh(C)_{/B} \xrightarrow{\ell_B} \mathcal{E}_{/\ell(B)}$ pres products

have univ colim, descent

prod of rep

$$\begin{tikzcd} P \arrow[r] \arrow[d] & p(c_2) \arrow[d] \\ p(c_1) \arrow[r] & p(c) \end{tikzcd}$$

$\longleftarrow$ Lemma

General case: $\ell : \mathrm{Psh}(C) \longrightarrow \mathcal{E}$ colim preserving

$\in \mathrm{Fun}(I, \mathrm{Psh}(C))$

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ X \arrow[r] & B \end{tikzcd} \quad \overset{\ell}{\Longrightarrow} \quad \text{p.b?}$$

Write $B \simeq \mathrm{colim}_{i \in I} B_i$, B_i representable. $\in \mathrm{Psh}(C)$

$\underline{i \in I}$

$\Longrightarrow$ Pull back to (along $B_i \to B$)

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X'_i \arrow[d] \\ X_i \arrow[r] & B_i \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d] \\ X \arrow[r] & B \end{tikzcd} \in \mathrm{Fun}(I, \mathrm{PSh}(C)) \quad \rightsquigarrow \quad \begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X'_i \arrow[d] \\ X_i \arrow[r] & B_i \end{tikzcd}$$

$\mathrm{PSh}(C)$

$\leftarrow$ rep

$B = \mathrm{colim}\, B_i \implies$ $P = \mathrm{colim}\, P_i$, $X = \mathrm{colim}\, X_i$, $X' = \mathrm{colim}\, X'_i$

colims are univ in $\mathrm{PSh}(C)$

and $\forall\, i \to j$ in I

$$\begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_i \arrow[d] \\ P_j \arrow[r] & X_j \arrow[r] & B_j \end{tikzcd} \qquad \begin{tikzcd} P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X'_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B_i \arrow[d] \\ P_j \arrow[r] & X'_j \arrow[r] & B_j \end{tikzcd}$$

$\leftarrow$ rep $\qquad \leftarrow$ rep

pb in $\mathrm{PSh}(C)$

ℓ preserves these pullbacks over a representable (special case!)

$$\begin{tikzcd}
\ell P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \ell X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \ell B_i \arrow[d] \\
\ell P_j \arrow[r] & \ell X_j \arrow[r] & \ell B_j
\end{tikzcd}$$

cart nat transf

and

$\ell P = \mathrm{colim}\, \ell P_i$

$\ell X = \mathrm{colim}\, \ell X_i$

$\ell B = \mathrm{colim}\, \ell B_i$

ℓ colim

$I \to \mathrm{Cart}(\mathcal{E}^{\to}) \to \mathcal{E}^{\to}$

descent

$$\begin{tikzcd}
\ell P_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \ell X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \ell B_i \arrow[d] \\
\ell P \arrow[r] & \ell X \arrow[r] & \ell B
\end{tikzcd}$$

$\to$ pullback of $\ell P \xrightarrow{f} \ell X \times_{\ell B} \ell X'$ along each $\ell B_i \to \ell B$ is iso

$\Rightarrow$ univ colim in $\mathcal{E}$ $\Rightarrow$ f iso

$\ell P_i \to \ell X_i \times_{\ell B_i} \ell X'_i$

Next time:

- "Object classifier"
- Truncation + Connectivity